Trygonometry Identities

$2\cos\theta\cos\varphi = \cos(\theta-\varphi)+\cos(\theta+\varphi)$

$\sin(\theta\pm\varphi) = \sin\theta\cos\varphi \pm \sin\varphi\cos\theta$

$\cos(\theta\pm\varphi) = \cos\theta\cos\varphi \mp \sin\varphi\sin\theta$

$2i\sin(x) = e^{ix}-e^{-ix}, \quad 2\cos(x) = e^{ix}+e^{-ix}$

Green's Identities

(I) $\int_\Omega (u\Delta v + \nabla u\cdot\nabla v)\,dx = \oint_{\partial\Omega} u\frac{\partial v}{\partial n}\,ds$

(II) $\int_\Omega (u\Delta v - v\Delta u)\,dx = \oint_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds$

Poisson Integral Formula

Th^m: The solution to

$\Delta u = 0 \quad x^2+y^2<1$ and $u=h \quad x^2+y^2=1$ is

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\,\frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)}\,d\phi$$

The Wave Eq: D'Alembert's Formula

Th: The solution to the IVP

$u_{tt} = c^2u_{xx}, \quad u(0,x)=f(x),\ u_t(0,x)=g(x) \quad x\in\mathbb{R}$

is given by

$$u(t,x) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz$$

Th: The solution to the IVP

$u_{tt} = c^2u_{xx}+F(t,x), \quad u(0,x)=f(x),\ u_t(t,x)=g(x),\ x\in\mathbb{R},\ t>0$

is given by

$$u(t,x) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy + \frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)} F(s,y)\,dy\,ds$$

Delta Function

Def: If $g_n(x)\to 0$ as $n\to\infty$ $\quad x\neq\xi$ and

$\int_a^b g_n(x)\,dx = 1 \quad \forall n$ then $\delta_\xi = \lim_{n\to\infty} g_n(x)$

Def: Unit step function

$$\sigma_\xi(x) = \begin{cases} 0 & x<\xi \\ 1 & x>\xi \end{cases}$$

Def: Ramp function

$$\rho(x-\xi) = \begin{cases} 0 & x<\xi \\ x-\xi & x>\xi \end{cases}$$

Ansatz 2nd Order ODE

$V''(x) - \lambda V(x) = 0$

$\lambda = \omega^2 > 0 \quad V(x) = Ae^{-\omega x} + Be^{\omega x}$

$\lambda = \omega^2 = 0 \quad V(x) = Ax+B$

$\lambda = -\omega^2 < 0 \quad V(x) = A\cos(\omega x) + B\sin(\omega x)$

Laplacian in Polar Coordinates

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

Fourier Series

Def: The F.S. of $f(x)$ on $-\pi\le x\le\pi$ is

$$\frac{a_0}{2} + \sum_{k=1}^\infty [a_k\cos kx + b_k\sin kx]$$

whose coef. are given by

$a_k = \langle f,\cos kx\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx \quad k=0,1,\ldots$

$b_k = \langle f,\sin kx\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx \quad k=1,2,\ldots$

Def: Complex F.S.

$f(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}$

where

$c_k = \langle f, e^{ikx}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx$

→ Change of Scale

Def: $[-l,l]$ the F.S. (with $x = l/\pi\, y$, $y\in[-\pi,\pi]$) is

$$\frac{a_0}{2} + \sum_{k=1}^\infty \left[a_k\cos\frac{k\pi x}{l} + b_k\sin\frac{k\pi x}{l}\right]$$

$$a_k = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{k\pi x}{l}\,dx$$

$$b_k = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{k\pi x}{l}\,dx$$

Fourier Transform

Def: $f(x)$ - defined on $\mathbb{R}$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}\,dk$$

Thm: The Fourier transform is linear

Th^m: If $f(x)$ has F.T $\hat{f}(k)$

$\Rightarrow f(x-\xi)$ has F.T $e^{-ik\xi}\hat{f}(k)$

$f(x)$	$\hat{f}(k)$
$\tan^{-1}x$	$\pi^{3/2}/\sqrt{2}\,\delta(k) - i\sqrt{\pi/2}\,e^{-\lvert k\rvert}/k$
$f(cx+d)$	$e^{ikd/c}/\lvert c\rvert\,\hat{f}(k/c)$
1	$\sqrt{2\pi}\,\delta(k)$
$\delta(x-\xi)$	$e^{-ix\xi}/\sqrt{2\pi}$
$\sigma(x)$	$\sqrt{\pi/2}\,\delta(k) - i/(\sqrt{2\pi}\,k)$
$\operatorname{sign} x$	$-i\sqrt{\pi/2}\;1/k$
$\sigma(x+a)-\sigma(x-a)$	$\sqrt{2/\pi}\,\sin(ak)/k$
$e^{-ax}\sigma(x)$	$1/[\sqrt{2\pi}(a+ik)]$
$e^{ax}(1-\sigma(x))$	$1/[\sqrt{2\pi}(a-ik)]$
$e^{-a\lvert x\rvert}$	$\sqrt{2/\pi}\,a/(k^2+a^2)$
e^{-ax^2}	$e^{-k^2/(4a)}/\sqrt{2a}$
$f(x)$	$\hat{f}(-k)$
$\hat{f}(x)$	$f(-k)$
$f'(x)$	$ik\hat{f}(k)$
$xf(x)$	$i\hat{f}'(k)$
$f*g(x)$	$\sqrt{2\pi}\,\hat{f}(k)\hat{g}(k)$
$f^{(n)}(x)$	$(ik)^n\hat{f}(k)$

Canonical Form Linear PDEs

$$L[u] = Au_{tt} + Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = G$$

Th: At a point (t,x), the linear 2nd order PDE is called

1. Hyperbolic if $\Delta(t,x) = B^2-4AC > 0$
   Canonical form: $L[u] = u_{xx} - u_{yy}$
2. Parabolic if $\Delta(t,x) = 0$ but $A^2+B^2+C^2\neq 0$
   $L[u] = \pm u_{xx}$
3. Elliptic if $\Delta(t,x) < 0$
   $L[u] = \pm(u_{xx}+u_{yy})$
4. Singular if $A=B=C=0$

Th: Linear transformation in canonical form

1. Determine the type
2. Set $\partial_x^2 = u_{xx}$, $\partial_y\partial_x = u_{xy}$, etc... in $L[u]$
3. Basic algebra to get $L[u]$ in canonical form.
4. Define the relative change of variables

The Free Space Green Function

Goal: Solve 2nd order PDE for arbitrary external forces (i.e. $-\Delta u = f$)

(1) Solve $-\Delta u = \delta(x-\xi) \to$ solution $G(x;\xi)$

(2) $f(x) = \int f(\xi)\,\delta(x-\xi)\,d\xi$

(3) $u(x) = \int f(\xi)\,G(x;\xi)\,d\xi$ (superposition principle)

Th^m: (Maximum) Let u be non constant harmonic on a bounded domain Ω and continuous on $\partial\Omega$. Then u achieves its max and min values only at boundary points

Def: Let Ω be a bounded and open connected domain. A function $u\in C^2(\Omega)\cap C(\bar\Omega)$ is said to be

1. subharmonic if $-\Delta u \le 0$ in Ω
2. superharmonic if $-\Delta u \ge 0$ in Ω
3. harmonic if $-\Delta u = 0$ in Ω

Standard Integration Tec.

Substitution: $u = g(x)$

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Integration by parts:

$$\int u\,dv = uv - \int v\,du$$

Heat Eq.

IVP: $u_t - ku_{xx} = 0,\ k>0,\ x\in\mathbb{R},\ u(0,x) = \phi(x)$

$$u(t,x) = 1/\sqrt{4\pi kt}\int_{-\infty}^{\infty}\exp\{-(x-\xi)^2/4kt\}\,\phi(y)\,dy$$

Energy: If $u(t,x)$ is a solution. Then

$$E(t) = \int_0^l u^2\,dx$$